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Self-dual Walker metrics with a two-step nilpotent Ricci operator

J. Davidov*, O. Muškarov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Street Bl.8, 1113 Sofia, Bulgaria

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Abstract

Motivated by the theory of hyperbolic twistor spaces, we obtain a local description of self-dual Walker metrics whose traceless Ricci operator, considered as a bundle-valued 2-form, is two-step nilpotent. The Einstein condition for Walker metrics is also discussed.
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1. Introduction

A neutral metric g (i.e. of split signature $(2, 2)$) on a 4-manifold M is said to be a Walker metric if there exists a two-dimensional null distribution on M, which is parallel with respect to the Levi-Civita connection of g. This type of metrics has been introduced by Walker [9] who has shown that they have a (local) canonical form depending on three smooth functions. Various curvature properties of some special classes of Walker metrics have been studied in [2,3,5,6] where several examples of neutral metrics with interesting geometric properties have been given. These include the non-flat Kähler–Einstein neutral metrics on complex tori and primary Kodaira surfaces constructed in [8].

In this note we study the $SO_0(2, 2)$-irreducible components [7] of the curvature tensor of the Walker metrics, where $SO_0(2, 2)$ is the identity component of $O(2, 2)$. In particular, we discuss the self-dual, anti-self-dual and Einstein conditions for these metrics. Moreover, we obtain a local description of the self-dual Walker metrics with constant scalar curvature whose traceless Ricci tensor $\mathcal{B}$, considered as a bundle-valued 2-form, has the property $\mathcal{B}^2|\Lambda_- = 0$, where Λ_- is the bundle of anti-self-dual bivectors. The motivation for considering such neutral metrics comes from the fact that they yield non-Kähler isotropic Kähler metrics [4] on the so-called hyperbolic twistor spaces [1]. The self-dual Walker metrics with a two-step nilpotent Ricci operator, i.e. $\mathcal{B}^2 = 0$, are discussed as well.

It should be noted that the local descriptions of self-dual and Einstein self-dual Walker metrics in Theorem 1 and Corollary 2 below have been also obtained in [3] where a local classification of a special class of neutral Osserman metrics has been given.

* Corresponding author. Tel.: +359 2 979 38 00; fax: +359 2 971 36 49.
E-mail addresses: jtd@math.bas.bg (J. Davidov), muskarov@math.bas.bg (O. Muškarov).

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2. Preliminaries

Let M be an oriented four-dimensional manifold with a neutral metric g, i.e. a metric of signature (2, 2). The metric g induces an inner product on the bundle Λ^2 of bivectors via

$$\langle X_1 \wedge X_2, X_3 \wedge X_4 \rangle = \frac{1}{2}[g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)],$$

$X_1, \ldots, X_4 \in TM$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_4$ be a local oriented orthonormal frame of TM with $\|\mathbf{e}_1\|^2 = \|\mathbf{e}_2\|^2 = 1$, $\|\mathbf{e}_3\|^2 = \|\mathbf{e}_4\|^2 = -1$. As in the Riemannian case, the Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is an involution given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Denote by $\Lambda_\pm$ the subbundles of Λ^2 determined by the eigenvalues ± 1 of the Hodge star operator. Set

$$\begin{aligned} s_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, \quad & \bar{s}_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \\ s_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4, \quad & \bar{s}_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, \\ s_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3, \quad & \bar{s}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3. \end{aligned} \tag{1}$$

Then $\{s_1, s_2, s_3\}$ and $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ are local oriented orthonormal frames of Λ_- and Λ_+ respectively with $\|s_1\|^2 = \|\bar{s}_1\|^2 = 1$, $\|s_2\|^2 = \|\bar{s}_2\|^2 = \|s_3\|^2 = \|\bar{s}_3\|^2 = -1$.

Let $\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$ be the curvature operator of (M, g). It is related to the curvature tensor R by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T); \quad X, Y, Z, T \in TM.$$

In this paper we adopt the following definition of the curvature tensor $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. The curvature operator $\mathcal{R}$ admits an $SO_0(2, 2)$-irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6} I + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$$

similar to that in the four-dimensional Riemannian case. Here τ is the scalar curvature, $\mathcal{B}$ represents the traceless Ricci tensor, $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ corresponds to the Weyl conformal tensor, and $\mathcal{W}_\pm = \mathcal{W}|\Lambda_\pm = \frac{1}{2}(\mathcal{W} \pm *\mathcal{W})$. The metric g is Einstein exactly when $\mathcal{B} = 0$ and is conformally flat when $\mathcal{W} = 0$. It is said to be *self-dual* (resp. *anti-self-dual*) if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$).

Recall that, by a result of Walker [9], for every Walker metric g on a 4-manifold M there exist local coordinates (x, y, z, t) around any point of M such that the matrix of g with respect to the frame $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})$ has the following form:

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \tag{2}$$

where a, b, c are smooth functions.

The components of the curvature tensor of g with respect to the frame $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})$ have been computed in [6] (see also [5]) and we shall make use of the formulas obtained there throughout the present paper.

3. The curvature operator of a Walker metric

Let g be a Walker metric on $\mathbb{R}^4$ having the form (2) with respect to the standard coordinates (x, y, z, t) of $\mathbb{R}^4$. Set

$$\begin{aligned} \mathbf{e}_1 &= \frac{1-a}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \quad & \mathbf{e}_2 &= \frac{1-b}{2}\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \\ \mathbf{e}_3 &= -\frac{1+a}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \quad & \mathbf{e}_4 &= -\frac{1+b}{2}\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}. \end{aligned} \tag{3}$$

Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is an oriented g-orthonormal frame of $T\mathbb{R}^4$.

Let $\{s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3\}$ be the frame of $\Lambda^2 = \Lambda_- \oplus \Lambda_+$ defined by means of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ via (1). Then

$$\begin{aligned} s_1 &= -\frac{a+b}{2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} \\ s_2 &= \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} \\ s_3 &= \frac{a-b}{2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \bar{s}_1 &= \frac{1+ab}{2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + 2c\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - a\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + b\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + 2\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t} \\ \bar{s}_2 &= c\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t} \\ \bar{s}_3 &= \frac{ab-1}{2}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + 2c\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - a\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + b\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + 2\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}. \end{aligned} \tag{5}$$

Next we give the matrix representations of the irreducible components of the curvature operator $\mathcal{R}$ with respect to the frame (4), (5).

3.1. *The anti-self-dual and self-dual Weyl operators*

Set

$$\mathcal{R}_{ij} = \langle \mathcal{R}(s_i), s_j \rangle, \quad i, j = 1, 2, 3.$$

Then the matrix of the anti-self-dual Weyl operator $\mathcal{W}_- : \Lambda_- \to \Lambda_-$ with respect to the frame $\{s_1, s_2, s_3\}$ has the form

$$\mathcal{W}_- = \begin{bmatrix} \mathcal{R}_{11} - \frac{\tau}{6} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ -\mathcal{R}_{12} & -\mathcal{R}_{22} - \frac{\tau}{6} & -\mathcal{R}_{23} \\ -\mathcal{R}_{13} & -\mathcal{R}_{23} & -\mathcal{R}_{33} - \frac{\tau}{6} \end{bmatrix}, \tag{6}$$

where τ is the scalar curvature.

Straightforward computations making use of (4) and the curvature formulas in [6] give

$$\begin{aligned} \mathcal{R}_{11} &= -\frac{1}{2}(b_{xx} + a_{yy} - 2c_{xy}) \\ \mathcal{R}_{12} &= -\frac{1}{2}(c_{xx} - b_{xy} - a_{xy} + c_{yy}) \\ \mathcal{R}_{13} &= -\frac{1}{2}(b_{xx} - a_{yy}) \\ \mathcal{R}_{22} &= -\frac{1}{2}(a_{xx} + b_{yy} - 2c_{xy}) \\ \mathcal{R}_{23} &= -\frac{1}{2}(c_{xx} + a_{xy} - b_{xy} - c_{yy}) \\ \mathcal{R}_{33} &= -\frac{1}{2}(b_{xx} + a_{yy} + 2c_{xy}), \end{aligned} \tag{7}$$

where subscripts in the right-hand side mean partial derivatives. Therefore for the scalar curvature τ we have

$$\tau = 2(\langle \mathcal{R}(s_1),\ s_1 \rangle - \langle \mathcal{R}(s_2),\ s_2 \rangle - \langle \mathcal{R}(s_3),\ s_3 \rangle) = a_{xx} + b_{yy} + 2c_{xy}. \tag{8}$$

In the next theorem we describe explicitly the self-dual Walker metrics (see also [3]).

Theorem 1. *A Walker metric is self-dual if and only if the functions a, b, c have the form*

$$
\begin{aligned}
a &= x^2yA + x^3B + x^2C + 2xyD + xE + yF + G,\\
b &= xy^2B + y^3A + y^2K + 2xyL + xM + yN + P,\\
c &= x^2yB + xy^2A + x^2L + y^2D + \frac{1}{2}xy(C + K) + xQ + yR + S,
\end{aligned} \tag{9}
$$

where A, B, C, etc. *are smooth functions depending only on z and t.*

Proof. Identities (6)–(8) imply that the self-duality condition for a Walker metric (2) is equivalent to the equations

$$a_{yy} = b_{xx} = 0, \quad a_{xy} = c_{yy}, \quad b_{xy} = c_{xx}, \quad a_{xx} + b_{yy} = 4c_{xy}. \tag{10}$$

Suppose that the functions a, b, c satisfy these equations. Then it is easy to check that all partial derivatives of a, b, c of order 4 with respect to x and y vanish. Therefore a, b, c are polynomials of degree 3 with respect to x and y with coefficients that are smooth functions of z and t. Now putting these polynomials into (10), one can easily see that the functions a, b, c must have the form (9). Conversely, if a, b, c have this form, it is trivial to check that they satisfy Eqs. (10). □

To write down the matrix representation of the self-dual Weyl operator $\mathcal{W}_+ : \Lambda_+ \to \Lambda_+$ with respect to the frame $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ we set

$$\mathcal{R}_{\bar{i}\bar{j}} = \langle \mathcal{R}(\bar{s}_i), \bar{s}_j \rangle, \quad i, j = 1, 2, 3.$$

Then making use of (5) and the curvature formulas in [6] we get

$$
\begin{aligned}
\mathcal{R}_{\bar{1}\bar{1}} = \mathcal{R}_{\bar{1}\bar{3}} = \mathcal{R}_{\bar{3}\bar{3}} = {} & -2c^2a_{xx} - \frac{1}{2}a^2b_{xx} - \frac{1}{2}b^2a_{yy} + 2acc_{xx} - 2bca_{xy} + abc_{xy} + 4ca_{xt} - 4cc_{xz} - 2ac_{xt}\\
& + 2ab_{xz} + 2ba_{yt} - 2bc_{yz} + 4c_{zt} - 2a_{tt} - 2b_{zz} + 2(a_xc_t - a_tc_x) + a_zb_x - a_xb_z + a_yb_t - a_tb_y\\
& + 2(b_yc_z - b_zc_y) + c(a_xb_y - a_yb_x) + a(b_xc_y - b_yc_x) + b(a_yc_x - a_xc_y),
\end{aligned} \tag{11}
$$

$$\mathcal{R}_{\bar{1}\bar{2}} = \mathcal{R}_{\bar{2}\bar{3}} = -ca_{xx} - cc_{xy} + \frac{1}{2}ac_{xx} + \frac{1}{2}ab_{xy} - \frac{1}{2}ba_{xy} - \frac{1}{2}bc_{yy} + a_{xt} - b_{yz} + c_{yt} - c_{xz}, \tag{12}$$

$$\mathcal{R}_{\bar{2}\bar{2}} = -\frac{1}{2}(a_{xx} + b_{yy} + 2c_{xy}). \tag{13}$$

This and (8) imply that

$$\mathcal{W}_+ = \begin{bmatrix} \mathcal{R}_{\bar{1}\bar{1}} - \frac{\tau}{6} & \mathcal{R}_{\bar{1}\bar{2}} & \mathcal{R}_{\bar{1}\bar{1}} \\ -\mathcal{R}_{\bar{1}\bar{2}} & \frac{\tau}{3} & -\mathcal{R}_{\bar{1}\bar{2}} \\ -\mathcal{R}_{\bar{1}\bar{1}} & -\mathcal{R}_{\bar{1}\bar{2}} & -\mathcal{R}_{\bar{1}\bar{1}} - \frac{\tau}{6} \end{bmatrix}. \tag{14}$$

In particular, any anti-self-dual Walker metric is scalar flat. We refer the reader to [3] for an analysis of the Jordan form of the operator $\mathcal{W}_+$.

Theorem 2. *A Walker metric is conformally flat if and only the functions a, b, c have the form*

$$
\begin{aligned}
a &= x^2C + 2xyD + xE + yF + G,\\
b &= -y^2C + 2xyL + xM + yN + P,\\
c &= x^2L + y^2D + xQ + yR + S,
\end{aligned}
$$

where C, D, E, etc. *are smooth functions of z and t obeying the following equations:*

$$
\begin{aligned}
& C_t - 2L_z = CQ - LE + DM,\\
& C_z + 2D_t = CR - LF + ND,\\
& E_t - N_z + R_t - Q_z = 2CS - 2LG + 2DP\\
& -2(PC_z + CP_z) + NQ_z + QN_z + 4(SL_z + LS_z) + EM_z + ME_z - 2(MR_z + RM_z) - NN_z - 3QQ_z
\end{aligned}
$$

$$
\begin{aligned}
&+NR_t+QE_t+FM_t+2MF_t+2SC_t+4CS_t-QR_t\\
&-4GL_t-6LG_t+2DP_t+4Q_{zt}-2E_{tt}-2M_{zz}=0,\\
&-2(FQ_t+QF_t)+NF_t+FN_t\ +4(SD_t+DS_t)+ER_t+RE_t+2(GC_t+CG_t)-EE_t-3RR_t\\
&-2SC_z-4CS_z+EQ_z+RN_z-RQ_z+2FM_z+MF_z\\
&-4PD_z-6DP_z+DG_z+4R_{zt}-2F_{tt}-2N_{zz}=0,\\
&2(SE_t+ES_t)+2(SN_z+NS_z)+2PF_t+FP_t+2GM_z+MG_z-2(PR_z+RP_z)-2(GQ_t+QG_t)\\
&-2SQ_z-2SR_t-EP_z-NG_t+4S_{zt}-2G_{tt}-2P_{zz}\\
&+S(EN-FM)+G(MR-NQ)+P(FQ-ER)=0.
\end{aligned}
$$

Proof. It follows from (10)–(14) that a Walker metric is conformally flat if and only if

$$
\begin{aligned}
&a_{yy}=b_{xx}=0,\quad a_{xx}+b_{yy}=0,\quad a_{xy}=c_{yy},\quad b_{xy}=c_{xx},\quad c_{xy}=0,\\
&ca_{xx}-ab_{xy}+ba_{xy}-a_{xt}+b_{yz}-c_{yt}+c_{xz}=0,\\
&2ca_{xt}+2cb_{yz}+2ab_{xz}+2ba_{yt}-2cc_{xz}-2ac_{xt}-2cc_{yt}-2bc_{yz}+4c_{zt}-2a_{tt}-2b_{zz}\\
&\quad+2(a_xc_t-a_tc_x)+2(b_yc_z-b_zc_y)+(a_zb_x-a_xb_z)+(a_yb_t-a_tb_y)\\
&\quad+c(a_xb_y-a_yb_x)+a(b_xc_y-b_yc_x)+b(a_yc_x-a_xc_y)=0.
\end{aligned}
$$

Now the result follows on plugging the expressions (9) for a, b, c into the above equations and comparing the coefficients of the variables x and y. □

3.2. The Ricci operator

It follows from [6] that the (1, 1)-tensor $\widehat{Ric}$ corresponding to the (2, 0)-Ricci tensor of a Walker metric (2) is given by

$$
\begin{aligned}
\widehat{Ric}\left(\frac{\partial}{\partial x}\right)&=\frac{1}{2}(a_{xx}+c_{xy})\frac{\partial}{\partial x}+\frac{1}{2}(b_{xy}+c_{xx})\frac{\partial}{\partial y},\\
\widehat{Ric}\left(\frac{\partial}{\partial y}\right)&=\frac{1}{2}(a_{xy}+c_{yy})\frac{\partial}{\partial x}+\frac{1}{2}(b_{yy}+c_{xy})\frac{\partial}{\partial y},\\
\widehat{Ric}\left(\frac{\partial}{\partial z}\right)&=\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\frac{1}{2}(a_{xx}+c_{xy})\frac{\partial}{\partial z}+\frac{1}{2}(a_{xy}+c_{yy})\frac{\partial}{\partial t},\\
\widehat{Ric}\left(\frac{\partial}{\partial t}\right)&=\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}+\frac{1}{2}(b_{xy}+c_{xx})\frac{\partial}{\partial z}+\frac{1}{2}(b_{yy}+c_{xy})\frac{\partial}{\partial t},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
2\alpha&=ca_{xy}+ba_{yy}-2a_{yt}-cc_{yy}-a_yc_x-c_y^2-ac_{xy}+2c_{yz}+c_ya_x+a_yb_y,\\
2\beta&=a_{xt}+b_{yz}-a_yb_x-ba_{xy}+cc_{xy}+c_xc_y-c_{yt}-c_{xz}+ac_{xx}-ca_{xx},\\
2\gamma&=a_{xt}+b_{yz}-a_yb_x-ab_{xy}+c_xc_y+bc_{yy}-cb_{yy}-c_{xz}+cc_{xy}-c_{ty},\\
2\delta&=ab_{xx}-2b_{xz}+a_xb_x+cb_{xy}-bc_{xy}-b_xc_y+c_xb_y-c_x^2-cc_{xx}+2c_{xt}.
\end{aligned}
\tag{16}
$$

Formulas (15) and (8) imply that the traceless Ricci tensor $Z=\widehat{Ric}-\frac{\tau}{4}Id$ is given by

$$
\begin{aligned}
Z\left(\frac{\partial}{\partial x}\right)&=\lambda\frac{\partial}{\partial x}+\mu\frac{\partial}{\partial y},\\
Z\left(\frac{\partial}{\partial y}\right)&=\nu\frac{\partial}{\partial x}-\lambda\frac{\partial}{\partial y},\\
Z\left(\frac{\partial}{\partial z}\right)&=\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\lambda\frac{\partial}{\partial z}+\nu\frac{\partial}{\partial t},\\
Z\left(\frac{\partial}{\partial t}\right)&=\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}+\mu\frac{\partial}{\partial z}-\lambda\frac{\partial}{\partial t},
\end{aligned}
\tag{17}
$$

where $\alpha, \beta, \gamma, \delta$ are defined by (16) and

$$4\lambda = a_{xx} - b_{yy}, \quad 2\mu = b_{xy} + c_{xx}, \quad 2\nu = a_{xy} + c_{yy}. \tag{18}$$

We have $\mathcal{B}(X \wedge Y) = Z(X) \wedge Y + X \wedge Z(Y)$ and formulas (4), (5) and (17) imply that

$$\begin{aligned}
\mathcal{B}(s_1) &= (\delta + \alpha + c(\nu - \mu))\bar{s}_1 + (\mu - \nu)\bar{s}_2 - (\delta + \alpha + c(\nu - \mu))\bar{s}_3,\\
\mathcal{B}(s_2) &= (\beta + \gamma - 2\lambda c)\bar{s}_1 + 2\lambda \bar{s}_2 - (\beta + \gamma - 2\lambda c)\bar{s}_3,\\
\mathcal{B}(s_3) &= (\delta - \alpha - c(\nu + \mu))\bar{s}_1 + (\mu + \nu)\bar{s}_2 - (\delta - \alpha - c(\nu + \mu))\bar{s}_3,\\
\mathcal{B}(\bar{s}_1) &= (\delta + \alpha + c(\nu - \mu))s_1 - (\beta + \gamma - 2\lambda c)s_2 - (\delta - \alpha - c(\nu + \mu))s_3,\\
\mathcal{B}(\bar{s}_2) &= -(\mu - \nu)s_1 + 2\lambda s_2 + (\mu + \nu)s_3,\\
\mathcal{B}(\bar{s}_3) &= (\delta + \alpha + c(\nu - \mu))s_1 - (\beta + \gamma - 2\lambda c)s_2 - (\delta - \alpha - c(\nu + \mu))s_3.
\end{aligned} \tag{19}$$

The Einstein condition is equivalent to the vanishing of the tensor Z and formulas (17), (16) and (18) imply the following result (see also [6]).

Theorem 3. *A Walker metric is Einstein if and only if*

$$a_{xx} = b_{yy}, \quad a_{xy} + c_{yy} = 0, \quad b_{xy} + c_{xx} = 0, \tag{20}$$

$$ba_{yy} + 2ca_{xy} - ac_{xy} - 2a_{yt} + 2c_{yz} + a_y b_y + a_x c_y - a_y c_x - c_y^2 = 0, \tag{21}$$

$$ab_{xy} + ba_{xy} + ca_{xx} - cc_{xy} - a_{xt} - b_{yz} + c_{yt} + c_{xz} + a_y b_x - c_x c_y = 0, \tag{22}$$

$$ab_{xx} + 2cb_{xy} - bc_{xy} - 2b_{xz} + 2c_{xt} + a_x b_x - b_x c_y + c_x b_y - c_x^2 = 0. \tag{23}$$

Corollary 1. *A Walker metric with $c = 0$ is Einstein if and only if the functions a and b have the form*

$$\begin{aligned}
a &= x^2 K + xA(z,t) + M(y,z,t),\\
b &= y^2 K + yB(z,t) + N(x,z,t),
\end{aligned} \tag{24}$$

where K is a constant and A, B, M, N are smooth functions satisfying the following PDE's:

$$N_x M_y = A_t + B_z, \tag{25}$$

$$[N_x(x^2 K + xA + M)]_x = 2N_{zx}, \tag{26}$$

$$[M_y(y^2 K + yB + N)]_y = 2M_{ty}. \tag{27}$$

Proof. Suppose that a Walker metric with $c = 0$ is Einstein. Then Eqs. (20) imply that the derivatives a_x and b_y have the form $a_x = \alpha(x,z,t)$, $b_y = \beta(y,z,t)$, where α and β are smooth functions for which $\alpha_x = \beta_y$. It is clear that the functions α_x and β_y depend only on the variables z and t; therefore we can write

$$a_x = 2xK(z,t) + A(z,t), \quad b_y = 2yK(z,t) + B(z,t)$$

for some smooth functions K, A, B. These identities imply that a and b have the form

$$a = x^2 K(z,t) + xA(z,t) + M(y,z,t), \quad b = y^2 K(z,t) + yB(z,t) + N(x,z,t),$$

where M and N are smooth functions. The scalar curvature of the given metric is constant and we infer from (8) that the function $K(z,t)$ is constant. This proves (24).

For $c = 0$, Eqs. (21) and (23) take the form $(ba_y)_y = 2a_{ty}$ and $(ab_x)_x = 2b_{zx}$. In view of (24), the latter equations imply (26) and (27), respectively. Moreover, it follows from (20) and (22) that $a_{xt} + b_{yz} = a_y b_x$ and, using (24), we obtain Eq. (25). □

Remark. Let us note that the description of the Einstein condition for the Walker metrics with $c = 0$ given in [2, Theorem 3] is incomplete since only the case when the functions N_x and M_y do not depend on the variables x and y, respectively, is considered. The next two examples show that, in general, N_x (resp. M_y) may depend on x (resp. y).

Example 1. In the case $b = 0$ and $c = 0$ Eqs. (25)–(27) are equivalent to the equations $A_t = 0$, $M_{ty} = 0$. Hence a Walker metric with $b = c = 0$ is Einstein if and only if $A = A(z)$ and $M = P(y, z) + Q(z, t)$, where A, P, Q are arbitrary smooth functions. In this case $a = xA(z) + P(y, z) + Q(z, t)$ and (8) implies that the metric is Ricci flat.

Example 2. Let K be a non-zero constant and let $A(z)$, $P(z)$ be arbitrary smooth functions. Set

$$a = x^2 K + xA(z) + P(z) \arctan y, \quad b = K(y^2 + 1), \quad c = 0.$$

Then it is easy to check that the functions a, b, c satisfy Eqs. (25)–(27); hence the corresponding Walker metric is Einstein with non-zero scalar curvature equal to $2K$.

Theorem 1 and identities (16)–(18) imply the following

Corollary 2. *A Walker metric is Einstein and self-dual if and only if the functions a, b, c have the form*

$$\begin{aligned} a &= x^2 K + xE(z, t) + yF(z, t) + G(z, t), \\ b &= y^2 K + xM(z, t) + yN(z, t) + P(z, t), \\ c &= xyK + xQ(z, t) + yR(z, t) + S(z, t), \end{aligned} \tag{28}$$

where K is a constant and E, F, G, etc. *are smooth functions satisfying the equations*

$$\begin{aligned} &2R_z - 2F_t = FQ + R^2 + KG - RE - FN, \\ &E_t + N_z - R_t - Q_z = FM - QR + KS, \\ &2Q_t - 2M_z = MR + Q^2 + KP - EM - QN. \end{aligned}$$

4. Walker metrics with $\mathcal{B}^2|\Lambda_- = 0$

The condition treated here appears when analyzing isotropic Kähler metrics on hyperbolic twistor spaces [1].

Theorem 4. *A Walker metric satisfies the condition $\mathcal{B}^2|\Lambda_- = 0$ if and only if*

$$a_{xx} = b_{yy}, \quad a_{xy} + c_{yy} = b_{xy} + c_{xx} = 0.$$

Proof. It follows from (19) that

$$\begin{aligned} \langle \mathcal{B}^2(s_1), s_1 \rangle &= \langle \mathcal{B}(s_1), \mathcal{B}(s_1) \rangle = -(\mu - \nu)^2, \\ \langle \mathcal{B}^2(s_2), s_2 \rangle &= \langle \mathcal{B}(s_2), \mathcal{B}(s_2) \rangle = -4\lambda^2, \\ \langle \mathcal{B}^2(s_3), s_3 \rangle &= \langle \mathcal{B}(s_3), \mathcal{B}(s_3) \rangle = -(\mu + \nu)^2. \end{aligned} \tag{29}$$

Therefore if $\mathcal{B}^2|\Lambda_- = 0$, then $\lambda = \mu = \nu = 0$. Conversely, if $\lambda = \mu = \nu = 0$, then by (19) we have $\langle \mathcal{B}^2(s_i), s_j \rangle = 0$ for $1 \le i, j \le 3$. Now the theorem follows from (18).

Next we consider the condition for the Ricci operator $\mathcal{B}$ to be two-step nilpotent.

Theorem 5. *A Walker metric satisfies the condition $\mathcal{B}^2 = 0$ if and only if*

$$a_{xx} = b_{yy}, \quad a_{xy} + c_{yy} = b_{xy} + c_{xx} = 0 \quad \text{and} \quad \alpha\delta = \beta^2, \tag{30}$$

where α, β, δ are the functions defined by (16).

Proof. We have $\mathcal{B}^2(X \wedge Y) = Z^2(X) \wedge Y + X \wedge Z^2(Y) + 2Z(X) \wedge Z(Y)$.

Suppose that $\mathcal{B}^2 = 0$. Then Theorem 4 implies that $\lambda = \mu = \nu = 0$ (the functions λ, μ, ν being defined by (18)). Therefore the functions a, b, c satisfy the equations stated in the theorem. Moreover, it follows from (17) that $Z^2 = 0$; thus $Z(X) \wedge Z(Y) = 0$ for all tangent vectors X, Y. The latter condition is equivalent to the identity $\alpha\delta = \beta\gamma$ as one can see by means of (17). We have $\beta = \gamma$, since $\lambda = \mu = \nu = 0$, thus $\alpha\delta = \beta^2$.

Conversely, if Eqs. (30) are satisfied, then (17) implies that $\mathcal{B}^2 = 0$. □

5. Self-dual Walker metrics with $\mathcal{B}^2|\Lambda_- = 0$

By a result of [1] the hyperbolic twistor space of a neutral 4-manifold is isotropic Kähler if and only if the metric is self-dual, $\mathcal{B}^2|\Lambda_- = 0$, and the scalar curvature is constant. Theorems 1 and 4 imply the following explicit description of the Walker metrics having these properties.

Theorem 6. *A Walker metric satisfies the conditions $\mathcal{W}_- = 0$ and $\mathcal{B}^2|\Lambda_- = 0$ if and only if the functions a, b, c have the form*

$$\begin{aligned} a &= x^2K(z,t) + xE(z,t) + yF(z,t) + G(z,t),\\ b &= y^2K(z,t) + xM(z,t) + yN(z,t) + P(z,t),\\ c &= xyK(z,t) + xQ(z,t) + yR(z,t) + S(z,t), \end{aligned} \tag{31}$$

where $K, E, F,$ etc. are arbitrary smooth functions. In this case the metric has constant scalar curvature if and only if $K(z,t) =$ const.

Theorems 5 and 6 together with (16) lead to

Corollary 3. *The conditions $\mathcal{W}_- = 0$ and $\mathcal{B}^2 = 0$ hold if and only if a, b, c have the form (31) with $K(z,t) \equiv const$ and*

$$\begin{aligned} &(RE + FN - KG - R^2 - FQ + 2R_z - 2F_t)(QN - RM + EM - Q^2 - KP + 2Q_t - 2M_z)\\ &\quad = (QR - FM - KS + E_t + N_z - R_t - Q_z)^2. \end{aligned}$$

In particular, any Walker metric with $\mathcal{W}_- = 0$, $\mathcal{B}^2 = 0$ has constant scalar curvature.

Proof. It follows from Theorems 5 and 6 that the conditions $\mathcal{W}_- = 0$, $\mathcal{B}^2 = 0$ hold if and only if the functions a, b, c have the form (31) and the functions α, β, δ defined by (16) are subject to the relation $\alpha\delta = \beta^2$. Using (16) and (31) we get that

$$\begin{aligned} 2\alpha &= 2xK_z - 2F_t + 2R_z + FN + ER - FQ - GK - R^2,\\ 2\beta &= xK_t + yK_z + E_t + N_z - Q_z - R_t - FM - KS + QR,\\ 2\delta &= 2yK_t - 2M_z + 2Q_t + EM - KP - MR + NQ - Q^2. \end{aligned}$$

Comparing the coefficients of x^2 and y^2 on the both sides of the identity $\alpha\delta = \beta^2$ gives $K_z = K_t = 0$. This proves the result. □

Remark. We do not know of examples of neutral metrics with non-constant scalar curvature satisfying the conditions $\mathcal{W}_- = 0$, $\mathcal{B}^2 = 0$.

Example 3. All the examples of neutral metrics with $\tau =$ const, $\mathcal{W}_- = 0$ and $\mathcal{B}^2|\Lambda_- = 0$ constructed in [1] also satisfy the condition $\mathcal{B}^2 = 0$. The next example shows that this is not true in general.

Let K be a non-zero constant and let G, P, S be smooth functions of (z,t) such that $GP \neq S^2$. Set

$$a = x^2K + G(z,t), \quad b = y^2K + P(z,t), \quad c = xyK + S(z,t).$$

In this case we have $\tau =$ const, $\mathcal{W}_- = 0$, $\mathcal{B}^2|\Lambda_- = 0$ by Theorem 6 and $\mathcal{B}^2 \neq 0$ by Corollary 3. Moreover $\mathcal{W} \neq 0$ by Theorem 2.

Example 4. Let G and P be arbitrary smooth functions of (z,t) and E, F, M, N non-zero constants such that $EN = FM$. Set

$$a = xE + yF + G(z,t), \quad b = xM + yN + P(z,t), \quad c = 0.$$

Then we have $\mathcal{W} = 0$, $\tau = 0$, $\mathcal{B}^2 = 0$, but $\mathcal{B} \neq 0$. In particular, the sectional curvature of the metric is not constant.

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